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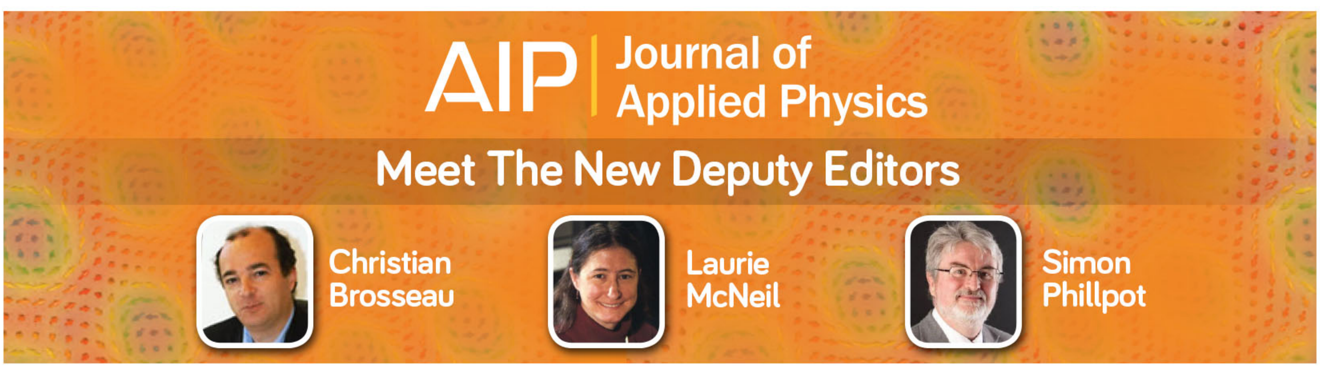

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Synchronization of pairwise-coupled, identical, relaxation oscillators based on metal-insulator phase transition devices: A model study

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Computing with networks of synchronous oscillators has attracted wide-spread attention as novel materials and device topologies have enabled realization of compact, scalable and low-power coupled oscillatory systems. Of particular interest are compact and low-power relaxation oscillators that have been recently demonstrated using MIT (metal-insulator-transition) devices using properties of correlated oxides. Further the computational capability of pairwise coupled relaxation oscillators has also been shown to outperform traditional Boolean digital logic circuits. This paper presents an analysis of the dynamics and synchronization of a system of two such identical coupled relaxation oscillators implemented with MIT devices. We focus on two implementations of the oscillator: (a) a D-D configuration where complementary MIT devices (D) are connected in series to provide oscillations and (b) a D-R configuration where it is composed of a resistor (R) in series with a voltage-triggered state changing MIT device (D). The MIT device acts like a hysteresis resistor with different resistances in the two different states. The synchronization dynamics of such a system has been analyzed with purely charge based coupling using a resistive (R_C) and a capacitive (C_C) element in parallel. It is shown that in a D-D configuration symmetric, identical and capacitively coupled relaxation oscillator system synchronizes to an anti-phase locking state, whereas when coupled resistively the system locks in phase. Further, we demonstrate that for certain range of values of R_C and C_C, a bistable system is possible which can have potential applications in associative computing. In D-R configuration, we demonstrate the existence of rich dynamics including non-monotonic flows and complex phase relationship governed by the ratios of the coupling impedance. Finally, the developed theoretical formulations have been shown to explain experimentally measured waveforms of such pairwise coupled relaxation oscillators.
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I. INTRODUCTION

Synchronization of systems of oscillators have attracted widespread attention among physicists, mathematicians, and neurobiologists alike. Even simple descriptions of oscillators and their coupling mechanisms give rise to rich dynamics. Synchronization dynamics of coupled oscillators not only have a wide variety of applications in engineering[1–3] but they also explain many natural, chemical and biological synchronization phenomena like the synchronized flashing of fireflies, pacemaker cells in the human heart, chemical oscillations, neural oscillations, and laser arrays, to name a few.[4] Coupled sinusoidal oscillators have been extensively studied[5–7] and their application in the computational paradigm has been well demonstrated.[8,9] A generalized description of oscillators in these models is usually a canonical phase model,[4,10] and the coupling mechanisms is generally assumed weak and composed of simple periodic functions. Several studies on more general periodic coupling functions have been studied.[11] Along with sinusoidal oscillators, non-linear Van-der-Pol oscillators and several of its variants have also been studied and the applicability of such models in neurobiological and chemical oscillators have been demonstrated.[12–15] Such analytic models of coupled oscillatory systems almost always require a canonical phase description of the oscillators and a periodic phase dependent additive coupling that can be classified as weak. Although such a description of a system of oscillators is elegant and provide key insights, relaxation oscillators that have recently been demonstrated using phase transition metal-insulator-transition (MIT) devices, cannot be modeled using such a simple phase description. Prior work by the authors have experimentally demonstrated locking and synchronization in a pairwise coupled system of relaxation oscillators[2] and its possible application in computation has also been discussed.[1] The coupling behavior of relaxation oscillators illustrate complex dynamical properties[16] and in this paper, we study the synchronization behavior of a pair of identical and electrically coupled relaxation oscillators.[2] Individual oscillators are composed of either two MIT devices in series (D-D configuration) or a MIT device in series with a linear resistor (D-R configuration)[2] and electrical coupling is enabled through a parallel connected R-C network. We show

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through analytical and numerical techniques how the final steady state relative phase of such coupled oscillators depend on the coupling function. For certain range of values of the coupling function, we note the possibility of a bistable system, where both in-phase and out-of-phase locking are stable, thereby giving rise to the possibility of using such oscillatory networks in computation.[1,9]

II. ELECTRICAL CIRCUIT MODEL AND REPRESENTATION

An electrical circuit representation of the relaxation oscillators is important to define the form of coupling which is physically realizable. The basic relaxation oscillator involves repeated charging and discharging of a capacitor through a resistive path. Switching between charging and discharging has to be done autonomously by the circuit configuration. In this paper, we are concerned with the relaxation oscillators built using state-changing devices. Such state-changing devices are fabricated using correlated oxide (vanadium dioxide, $V O_2$) and exhibit MIT where the device switches between a metallic and an insulating state under the application of heat or an electric field.[17] Further details about the physical implementation of these devices are discussed in Sec. IX. We will consider two kinds of relaxation oscillator circuits using such state-changing devices-(a) two state-changing devices in series (Figure 1). We will refer to this configuration as D-D. And (b) a state changing device in series with a resistance (Figure 2).[2] This configuration will be referred to as D-R. The D-D configuration is enticing in its simplicity, both in physical realization and analysis as will be evident in Secs. III and IV. The D-R configuration, on the other hand, has already been experimentally demonstrated[2] and can be thought of as an extension of the D-D configuration albeit with more complex dynamics of synchronization. In this paper, we will first study the D-D configuration, using analytical and numerical techniques; and show through phase models and flow analysis some key results in the D-R configuration.

The state transition of the device follows:

(a) Only the resistance of the device changes with its state; and the resistance is linear;
(b) A state transition is triggered by the voltage across the device. This triggering can be electric field driven or thermally driven, and can be modeled as an equivalent triggering voltage.[2] When the voltage exceeds a higher threshold v_h, the state changes to a metallic (low resistance) state and when the voltage exceeds a lower threshold v_l, the state switches back to the insulating (high resistance) state. The thresholds v_h and v_l are not equal, i.e., there is hysteresis in the switching with $v_l < v_h$, and
(c) A capacitance is associated with the device that ensures gradual build up and decaying of the voltage (and hence energy) across the device v_D.

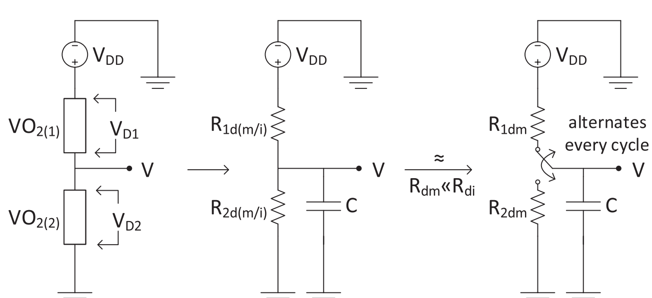

FIG. 1. Relaxation oscillator circuit realized using two MIT state-changing devices in series (D-D configuration), and its circuit equivalent with R_{dm} and R_{di} as the internal resistance of the MIT devices in metallic and insulating states, respectively. When $R_{di} \gg R_{dm}$ the device behaves as a parallel combination of a capacitor and a resistor with a switch.

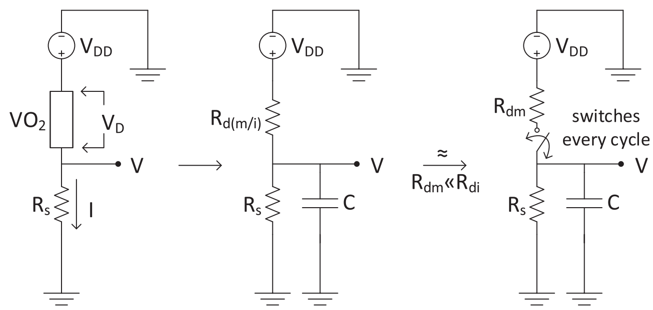

FIG. 2. Relaxation oscillator circuit realized with a MIT device in series with a resistor (D-R configuration), and its circuit equivalent with R_{dm} and R_{di} as the internal resistance of the MIT device in metallic and insulating states, respectively. When $R_{di} \gg R_{dm}$ the device behaves as a parallel combination of a capacitor and a resistor with a switch.

The present study of the synchronization dynamics of such coupled systems, although inspired by the experimental realization of VO_2 based oscillators, is not limited to these oscillators only, but encompasses a class of similar pairwise-coupled relaxation oscillators as well. The circuit equivalents of single and coupled relaxation oscillators are shown in Figs. 1–3, respectively. The internal resistance of the device R_d has two different values in the two states of the device-R_{di} in the insulating (high resistance) state and R_{dm} in the metallic (low resistance) state. C is the internal capacitance of the MIT device (including any parasitic capacitances) and R_S is the series resistance. We will also assume that $R_{di} \gg R_{dm}$. In the D-D configuration, the capacitor being charged can be

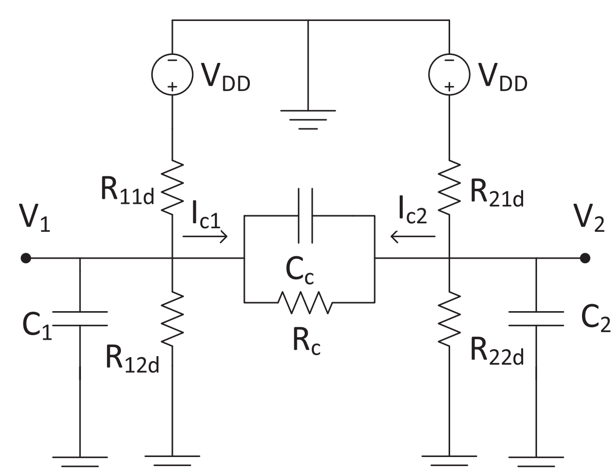

FIG. 3. Circuit equivalent of coupled D-D oscillators of with an RC circuit used as the coupling circuit. For D-R oscillators, one state-changing resistor of each oscillator is replaced by a constant linear resistor.

represented as a single capacitor at the output circuit node. The coupling circuit is a parallel combination of a capacitor C_c and a resistor R_c. As shown, the output node of the oscillator is between the device and the resistance, and the coupling circuit is connected between these output nodes.[2]

III. MODEL DEVELOPMENT FOR ISOLATED AND COUPLED OSCILLATORS

Before investigating the system dynamics, let us establish the system model and the system of differential equations that define the system. This will allow us to define the conditions for oscillation as well as the coupling dynamics. We will first consider D-D configuration and then D-R configuration as an extension of the D-D configuration. The D-D configuration, owing to its inherent symmetry renders to easier dynamics and analysis and provides valuable insights into the system. Such key numerical and analytical results for this are discussed in Secs. V, VI, and VII.

A. D-D configuration

The circuit equivalent for a D-D type relaxation oscillator is shown in Figure 1. For simplicity, all voltages are normalized to v_{dd} (including v_l and v_h). We define conductances $g_{di} = R_{di}^{-1}$, $g_{dm} = R_{dm}^{-1}$, and $g_c = R_C^{-1}$. For the conductances, subscript d denotes a state dependent device conductance and m/i denotes metallic/insulating state, respectively. The subscripts preceding dm or di refer to the corresponding numbered device as shown in figure. Also, it is assumed that $g_{dm} \gg g_{di}$, which means that the g_{di} state essentially disconnects the circuit. This implies that the effective charging happens through g_{1dm} and effective discharging through g_{2dm}. The single D-D oscillator can be described by the following set of piecewise linear differential equations:

$$cv' = \begin{cases} (v_{dd} - v)g_{1dm} & \textit{charging} \\ -v\, g_{2dm} & \textit{discharging}, \end{cases} \tag{1}$$

where c is the lumped capacitance of both devices along with the parasitics. The equation can be re-written as

$$cv' = -g(s)v + p(s), \tag{2}$$

where s denotes the conduction state of the device (0 for metallic and 1 for insulating) and $g(s)$ and $p(s)$ depend on the device conduction state s as follows:

$$g(s) = \begin{cases} g_{1dm}, & s = 0 \\ g_{2dm}, & s = 1, \end{cases} \tag{3}$$

$$p(s) = \begin{cases} g_{1dm}, & s = 0 \\ 0, & s = 1. \end{cases} \tag{4}$$

When two identical oscillators are coupled in a manner described in Figure 3, the system can be described by the following coupled equations:

$$c_1 v_1' = \begin{cases} (v_{dd} - v_1)g_{11dm} - i_{c1} & \textit{charging} \\ -v_1\, g_{12dm} - i_{c1} & \textit{discharging}, \end{cases} \tag{5}$$

$$c_2 v_2' = \begin{cases} (v_{dd} - v_2)g_{21dm} - i_{c2} & \textit{charging} \\ -v_2\, g_{22dm} - i_{c2} & \textit{discharging}, \end{cases} \tag{6}$$

where c_1 and c_2 are the lumped capacitances of the oscillators. For conductances g, the first subscript denotes the oscillator and the second denotes the device. $i_{c1} = -i_{c2}$ is the coupling current given by

$$i_{c1} = (v_1' - v_2')c_c + (v_1 - v_2)g_c. \tag{7}$$

When coupled, the system has 4 conduction states $s = s_1 s_2 \in \{00, 01, 10, 11\}$ corresponding to the 4 combinations of s_1 and s_2. Analogous to (2), the coupled system can be described in matrix form as

$$\begin{aligned} c_c F x'(t) &= -g_c A(s)x(t) + P(s), \\ x'(t) &= -\frac{g_c}{c_c} F^{-1} A(s)\big(x(t) - A^{-1}(s)P(s)\big), \end{aligned} \tag{8}$$

where $x(t) = (v_1(t),\ v_2(t))$ is the state variable at any time instant t. The 2×2 matrices F and $A(s)$, and vector $P(s)$ are given by

$$F = \begin{bmatrix} 1+\alpha_1 & -1 \\ -1 & 1+\alpha_2 \end{bmatrix}, \tag{9}$$

$$\begin{aligned} A(00) &= \begin{bmatrix} -\beta_{11}-1 & 1 \\ 1 & -\beta_{21}-1 \end{bmatrix}, \quad P(00) = \begin{bmatrix} \beta_{11} \\ \beta_{21} \end{bmatrix}, \\ A(10) &= \begin{bmatrix} -\beta_{12}-1 & 1 \\ 1 & -\beta_{21}-1 \end{bmatrix}, \quad P(10) = \begin{bmatrix} 0 \\ \beta_{21} \end{bmatrix}, \\ A(01) &= \begin{bmatrix} -\beta_{11}-1 & 1 \\ 1 & -\beta_{22}-1 \end{bmatrix}, \quad P(01) = \begin{bmatrix} \beta_{11} \\ 0 \end{bmatrix}, \\ A(11) &= \begin{bmatrix} -\beta_{12}-1 & 1 \\ 1 & -\beta_{22}-1 \end{bmatrix}, \quad P(11) = 0. \end{aligned} \tag{10}$$

Here, $\alpha_i = c_i/c_c$ is the ratio of the combined lumped capacitance of ith oscillator to the coupling capacitance c_c, and $\beta_{ij} = g_{ijdm}/g_c$ is the ratio of the metallic state resistance of jth device of ith oscillator, where $i \in \{1, 2\}$ and $j \in \{1, 2\}$. The fixed point in a conduction state s is given by $p_s = A^{-1}(s)P(s)$ and the matrix determining the flow (the *flow matrix* or the *velocity matrix*) is given by $\frac{g_c}{c_c}F^{-1}A(s)$ as can be seen in (8). In Sec. V, we analyze the steady state locking and synchronization dynamics of two such identical oscillators coupled with a parallel resistive and capacitive element as shown in Figure 3.

B. D-R configuration

The equivalent circuit for a D-R type relaxation oscillator is shown in Figure 2. As in the case of D-D configuration, voltages are normalized to v_{dd}. The conductances involved are $g_{di} = R_{di}^{-1}$, $g_{dm} = R_{dm}^{-1}$, $g_s = R_s^{-1}$, and $g_c = R_C^{-1}$. Effective charging happens through g_{dm} as in the previous case but there is an added leakage through g_s, whereas effective discharging happens only through g_s. Following the same methodology as in the D-D case, the equation for the single D-D oscillator dynamics can be written as:

$$cv' = \begin{cases} (v_{dd} - v)g_{dm} - v\,g_s & \textit{charging} \\ -v\,g_s & \textit{discharging}, \end{cases} \tag{11}$$

which can be re-written as

$$cv' = -g(s)v + p(s), \tag{12}$$

where

$$g(s) = \begin{cases} g_{dm} + g_s, & s = 0 \\ g_s, & s = 1, \end{cases} \tag{13}$$

$$p(s) = \begin{cases} g_{dm}, & s = 0 \\ 0, & s = 1, \end{cases} \tag{14}$$

and s denotes the conduction state of the system as before. In case of coupled D-R oscillators, arguments similar to the previous case lead to the same matrix equation as (8)

$$x'(t) = -\frac{g_c}{c_c} F^{-1} A(s)\big(x(t) - A^{-1}(s)P(s)\big), \tag{15}$$

where matrices F and P remain the same as before but matrix A changes to the following:

$$F = \begin{bmatrix} 1+\alpha_1 & -1 \\ -1 & 1+\alpha_2 \end{bmatrix}, \tag{16}$$

$$A(00) = \begin{bmatrix} -\beta_1 - \beta_{s1} - 1 & 1 \\ 1 & -\beta_2 - \beta_{s2} - 1 \end{bmatrix}, \quad P(00) = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},$$

$$A(10) = \begin{bmatrix} -\beta_{s1} - 1 & 1 \\ 1 & -\beta_2 - \beta_{s2} - 1 \end{bmatrix}, \quad P(10) = \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix},$$

$$A(01) = \begin{bmatrix} -\beta_1 - \beta_{s1} - 1 & 1 \\ 1 & -\beta_{s2} - 1 \end{bmatrix}, \quad P(01) = \begin{bmatrix} \beta_1 \\ 0 \end{bmatrix},$$

$$A(11) = \begin{bmatrix} -\beta_{s1} - 1 & 1 \\ 1 & -\beta_{s2} - 1 \end{bmatrix}, \quad P(11) = 0. \tag{17}$$

Here, $\beta_i = g_{idm}/g_c$ and $\beta_{si} = g_{si}$.

For all numerical simulations in the rest of the paper, the normalized values of v_l and v_h w.r.t v_{dd} are chosen to be 0.2 and 0.8, respectively.

IV. PHASE SPACE, FLOWS AND OSCILLATION CONDITIONS

A. Single Oscillators

A series arrangement of two MIT devices (D-D), or an MIT device and a resistor (D-R) will oscillate only when certain conditions are met. In case of two devices in series (D-D), the two devices must be in opposite conduction states (one metallic and the other insulating) all the time for oscillations to occur. If the threshold voltages v_l and v_h are same for the devices and the following condition holds:

$$v_l + v_h = V_{DD}, \tag{18}$$

and at $t = 0$ the devices are in different conduction states, then any time one device switches, the other will make the opposite transition as well. The basic mechanism of oscillations is as follows. The device in metallic state connects the circuit and charges (discharges) the output capacitor, and the other device in insulating state does not participate in the dynamics. As the capacitor charges, the voltage drop across the device in metallic state decreases and crosses the lower threshold v_l. At the same instant, the voltage drop across the other device in insulating state increases and crosses the higher threshold v_h because $v_{D1} + v_{D2} = V_{DD}$. The devices then switch states and the cycle continues. The devices can be conceived as a switch which is open in insulating state (ignoring any leakage in the insulating state) and closed in metallic state (Figure 1). If v_l and v_h deviate from (18), the devices will not switch at the same instant and oscillations will stop as the system settles to a stable point where both devices are in same state and the voltage of the output nodes remains at $V_{DD}/2$. This may require additional startup circuit in the system, which is trivial to integrate.

In D-R configuration, another set of conditions have to be met[18] which depend on the relative values of the device resistances in the two states (R_{dm} and R_{di}) and the series resistance (R_S). These conditions can be described using the phase diagram of the MIT device Figure 4. Lines with slopes r_i and r_m are the regions of operation of the device in insulating and metallic states, respectively. The intersection of these lines with the load line due to the series resistance gives the stable points of the system in the two states. For self-sustained oscillations, the stable points in each state should lie outside the region of operation, i.e., outside the

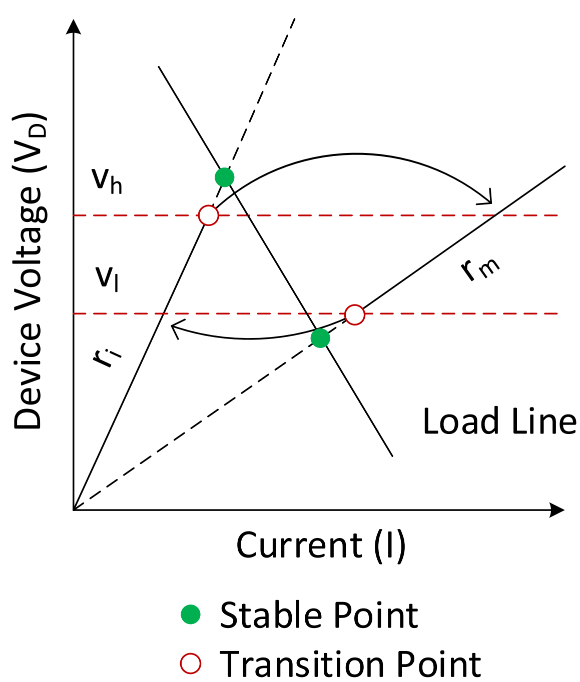

FIG. 4. Phase space of the device in a single D-R oscillator. Lines with slopes r_i and r_m are the regions of operation in insulating and metallic conduction states, respectively. The intersection of these regions with the load line are the stable points in each region of operation. The transition points should be encountered before reaching the stable points for sustained oscillations as shown.

region defined by horizontal lines passing through the transition points. This ensures that the system always tries to reach the stable point in the current state but is always preceded by a transition to the other state. This moves the system towards the stable point of the other state (away from the previous stable point) and hence the system never reaches any stable point and oscillates. This configuration is robust towards deviation of v_l and v_h from condition (18) and as only one device is involved, it does not require the difficult constraint of simultaneous switching of devices as was in the D-D case. This reduced requirement of symmetry is an attractive property of the D-R configuration as initial experiments have confirmed sustained oscillations in this configuration.[2]

We define the region of operation of a device (and hence of an oscillator) as the region where the device voltage lies between v_l and v_h (or the output voltage lies between $1-v_l$ and $1-v_h$). For the D-D case, the oscillators are expected to remain within the region of operation all the time. However, in the D-R case, the system can go outside the region of operation in a specific manner as described later.

B. Coupled oscillators

For analyzing the coupled circuits, the phase diagram of a coupled system can be drawn in the $v_1 \times i_1 \times v_2 \times i_2$ space as was done in Figure 4. However, we note that in a given conduction state of the system, $s = s_1 s_2$, (v_1, v_2) can uniquely identify the system, and hence, $v_1 \times v_2$ space is sufficient for a phase diagram. Therefore, we can draw 4 different phase diagrams of the system for each conduction state s (Figure 5) with transitions among them[16] (Figure 6). The transitions occur at the edges when either v_1 or v_2 reach the higher or lower threshold for state change from metallic to insulating or vice versa. The flows in each of the 4 conduction states are linear flows and hence have a single fixed point (Figure 5). The conditions for oscillations can be described using Figure 7. Analogous to the case of a single oscillator, these stable points should lie outside the region of operation (in the shaded region) in a way that the system always tries to move towards these stable points but should be preceded by a state transition which occurs when the system reaches the (red) dashed lines.

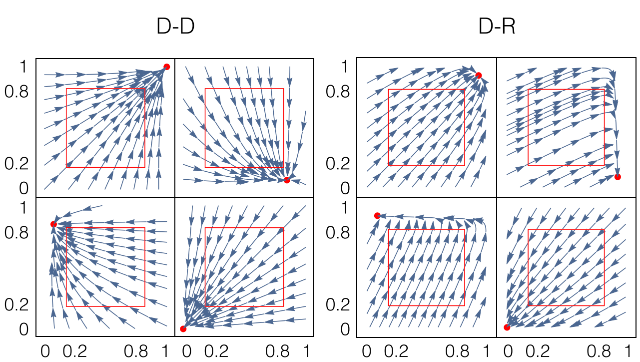

FIG. 5. The coupled system can be described by 4 different phase spaces for each state $s = s_1 s_2$. This figure shows the system flows of the D-D and D-R coupled oscillator system in the 4 regions of operation along with the fixed points (shown as red dots) in each state. This figure also represents the simplified case where the flows are monotonic within the region of operation.

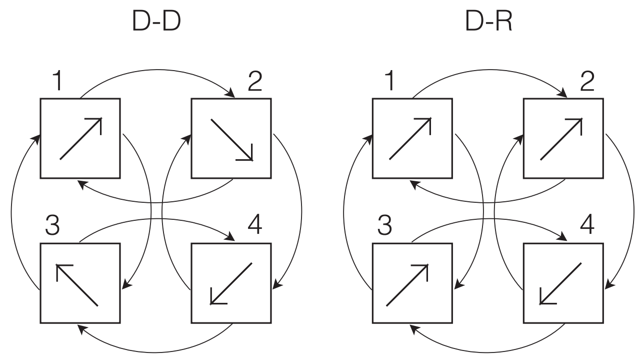

FIG. 6. Schematic representation showing the monotonic flow directions in the regions of operation in the simplified model. The monotonicity condition is sufficient for existence of a steady state periodic orbit in the D-D case. Transitions are shown among the 4 states 1(MM), 2(IM), 3(MI), and 4(II) of the coupled system when the system reaches any edge, i.e., the voltage of any oscillator reaches a phase change threshold of its MIT device.

1. Monotonic flows and periodic orbits

The conditions of Figure 7 are general enough to hold for both D-D and D-R configurations and they ensure that the system does not settle down to a stable point and voltages across oscillators repeatedly increase and decrease. However, these conditions do not ensure the existence of a stable orbit which can give periodic oscillations. To ensure existence of a stable periodic orbit, we consider additional conditions for the systems. For D-D configuration, we consider systems where the flows in the states are monotonic, i.e., v_1 and v_2 are either constantly increasing or constantly decreasing in the region of operation of any conduction state. Figure 6 show these monotonic directions with the state transitions for D-D coupled oscillator configurations. It is proved later that for two identical coupled D-D oscillators, this condition of monotonicity of the flows is *sufficient* for existence of a stable orbit and hence for periodic oscillations. For D-R coupled oscillators, we consider systems where either the direction of flows are strictly monotonic as shown in Figure

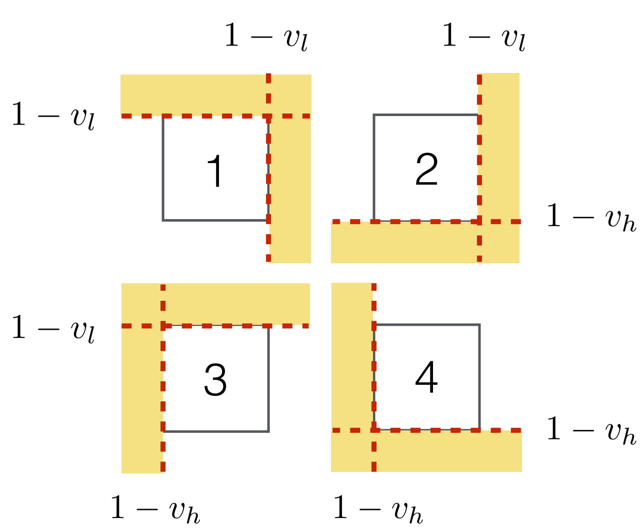

FIG. 7. The stable points of both D-D and D-R coupled oscillator system should lie in the yellow shaded region for the system to oscillate. The system undergoes a transition to another state when the system hits the red dashed lines.

6 or are non-monotonic in a very specific way as discussed in Sec. VII (see Figure 23). In this case, periodic oscillations can be ensured for certain conditions as described in Sec. VII. It should be noted here that in the D-R case, the system can also go outside the region of operation as seen in Figure 8(b), but if the fixed points lie in the above mentioned shaded regions, the system will always oscillate. Figure 8 shows typical time-domain waveforms and corresponding phase-space trajectories for the coupled oscillators of the D-D and D-R types.

V. SYMMETRIC D-D COUPLED OSCILLATOR DYNAMICS

Let us first investigate the case when the D-D oscillators are identical and their effective charging and discharging rates are equal, i.e., $\beta_{11} = \beta_{21} = \beta_{12} = \beta_{22} = \beta$ and $\alpha_1 = \alpha_2$. This corresponds to a well designed and ideal oscillator system where the pull-up and pull-down device resistances have been matched to create equal charging and discharging rates. In such a scenario the velocity matrices in the four conduction states $\frac{g_c}{c_c}F^{-1}A(s)$ become equal. As such, the state spaces in the four conduction states can be represented in a common state space with the system flow described by the common velocity matrix and a single fixed point. However, in this common state space, the regions of operation in the four conduction states will be four distinct regions. The position of these regions for a conduction state would depend on the position of its respective fixed points in the original state space. Such a combined phase space is shown in Figure 9.

The symmetry of the system is apparent in the flow as well. The eigen values λ_1, λ_2 and eigen vectors e_1, e_2 of the velocity matrix $\frac{g_c}{c_c}F^{-1}A$ of the symmetric system are

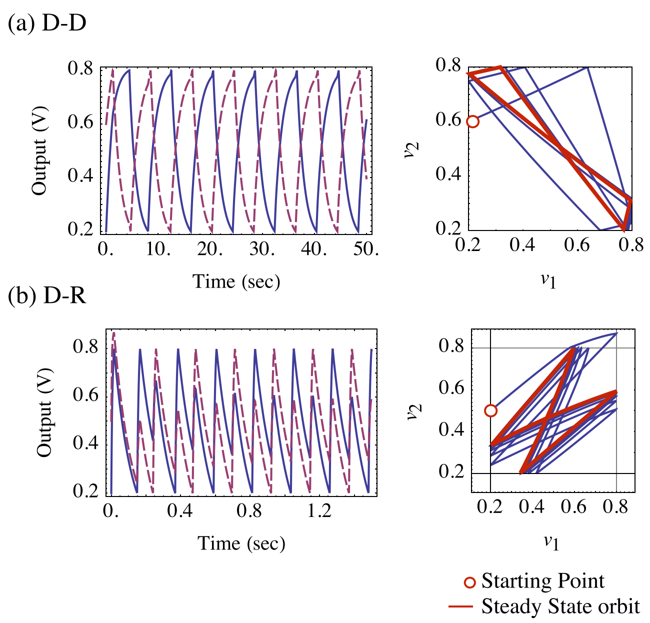

FIG. 8. Simulation waveforms with time (left) and the system trajectory in phase space (right) of a system of coupled oscillators of type (a) D-D and (b) D-R. The steady state periodic orbit is shown in red. The butterfly shaped steady state trajectory corresponds to waveforms similar to anti-phase locking. The solid and dashed lines represent output of the two oscillators.

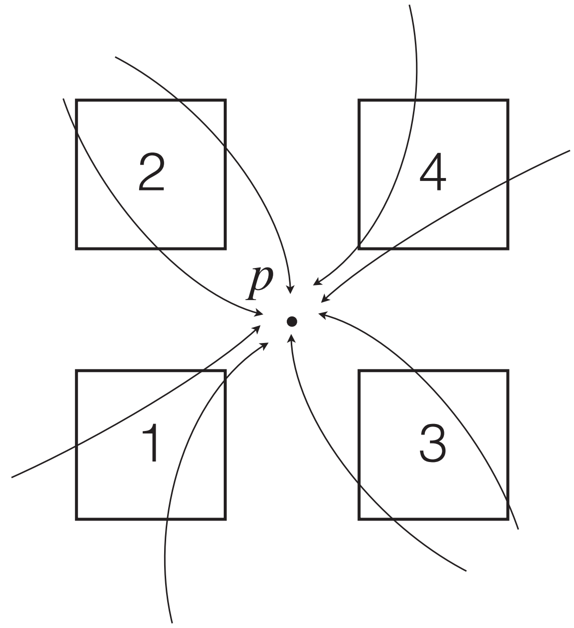

FIG. 9. Combined phase space in the symmetric D-D coupled oscillator case showing 4 regions of operation of the four different conduction states such that all states share a single fixed point p. This is possible as the flow matrices in all the four states are equal and, hence, all state spaces can be represented in a single space with a single flow but occupying different regions.

$$\lambda_1 = -\frac{g_c}{c_c}\left(\frac{\beta}{\alpha}\right), \quad \lambda_2 = -\frac{g_c}{c_c}\left(\frac{\beta+2}{\alpha+2}\right), \tag{19}$$

$$e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{20}$$

Real negative eigen values imply that the flow of the system is symmetric about both the eigen vector directions (i.e., a mirror image of itself about the eigen directions) as shown in Figure 10. The stable fixed points in the conduction states 1(00), 2(01), 3(10), and 4(11) are $p_1 = (1, 1)$,

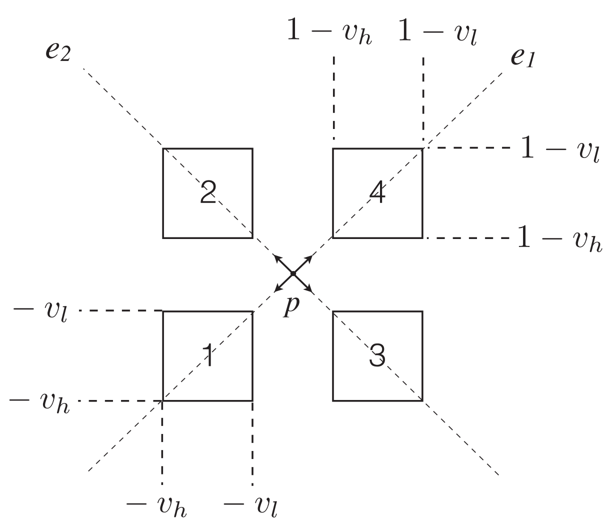

FIG. 10. In the combined phase space, the flow of the coupled system is a mirror image of itself about its eigen vector directions e_1 and e_2 as the eigen values are real and negative. This symmetry of the flows can be reduced and the state space of the system can be described by considering just one-fourth of this space as shown in Figure 11.

$p_2 = \left(1 - \frac{1}{2+\beta}, \frac{1}{2+\beta}\right)$, $p_3 = \left(\frac{1}{2+\beta}, 1 - \frac{1}{2+\beta}\right)$, and $p_4 = (0,\ 0)$, respectively. Hence, the line along the eigen vector e_1 is the diagonal for both conduction states 1 and 4. Under the assumption that the v_{dd} normalized thresholds v_l and v_h are symmetric, i.e., $v_l = 1 - v_h$, the line along e_2 also becomes the diagonal for states 2 and 3. This is because the fixed points of conduction states 2 and 3—p_2 and p_3 lie on $x + y = 1$ line in their original state spaces which is same as the eigen direction e_2. It should now be noted that the transitions between the conduction states, the regions of operation and the flow, all have the same common discrete symmetry—mirroring about e_1 and e_2. We can do a symmetry reduction at this point and the system can be completely described by just two states and two transitions (Figure 11(a)).

To study the steady state periodic orbits of this system, we calculate the return map on the left edge of state 1 in Figure 11(a) which is $f = f_1 \circ f_2$. In this case, any periodic orbit in the symmetry reduced space will correspond to at least one periodic orbit in the complete space (see Figure 13). Also, if no fixed point of the return map exist in the symmetry reduced space, then there is definitely no periodic orbit in the complete space. The coordinate measurements on the edges are defined as shown in Figure 11b. f_1: $x_1 \rightarrow y_1$ is the mapping from the left edge of state 1 to its top edge and f_2: $x_2 \rightarrow y_2$ is the mapping from left edge of state 2 to its bottom edge. x_1, x_2, x_3, and x_4 are defined on their respective edges as shown in Figure 11(b). As both the eigen values λ_1 and λ_2 are real and negative, $f_1(x)$ will lie above $x = y$ line and $f_2(x)$ will lie below it. A representative plot of f_1, f_2 and $f = f_1 \circ f_2$ (i.e., the return map f: $x_1 \rightarrow y_2$) is shown in Figure 12, where $dv = v_h – v_l$. The composition $f = f_1 \circ f_2$ lies above $x = y$ if f_2 is more curved than f_1 and vice versa. As the return map is always increasing, only the first return map needs to be considered for finding fixed points and the higher return maps do not add new fixed points. When the coupling is more capacitive, the composition function tends to be concave as shown in Figure 12(a). *Proposition* 1 gives a mathematical form to this notion where a sufficient

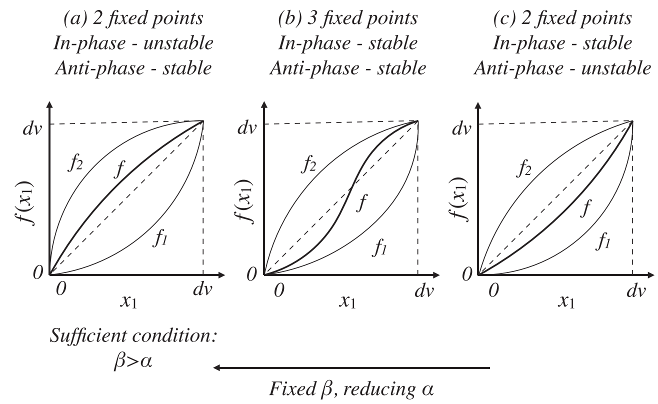

FIG. 12. Representative plot of mappings f_1, f_2 and their composition $f = f_1 \circ f_2$ with fixed β and varying α. Here, $dv = v_h – v_l$. $\beta > \alpha$ is a sufficient condition for a concave f and hence stable anti-phase locking. As α increases, the curve for f transitions into a s-shaped curve with both in-phase and anti-phase lockings stable, and then finally to a convex curve with stable in-phase locking.

condition is proved for anti-phase locking. We show numerically with simulations in Figure 15 the conditions for in-phase locking compared to anti-phase locking.

If the system moves from any arbitrary point on the flow, say (x_a, y_a) to another point, (x_b, y_b) in time t then the following implicit equation can be written:

$$\left(\frac{x_a + y_a}{x_b + y_b}\right)^{\frac{1}{\lambda_1}} = \left(\frac{x_a - y_a}{x_b - y_b}\right)^{\frac{1}{\lambda_2}}. \tag{21}$$

In state 1, (x_a, y_a) lies on the left edge and (x_b, y_b) lies on the top edge. To define f_1: $x_1 \rightarrow y_1$, we substitute $(x_a, y_a) = (-v_h, -v_h + x_1)$ in (21) and obtain an implicit equation for f_1 as

$$\left(\frac{2v_h - x_1}{2v_l + y_1}\right) = \left(\frac{x_1}{y_1}\right)^{\frac{\alpha+2\beta}{\beta+2\alpha}}. \tag{22}$$

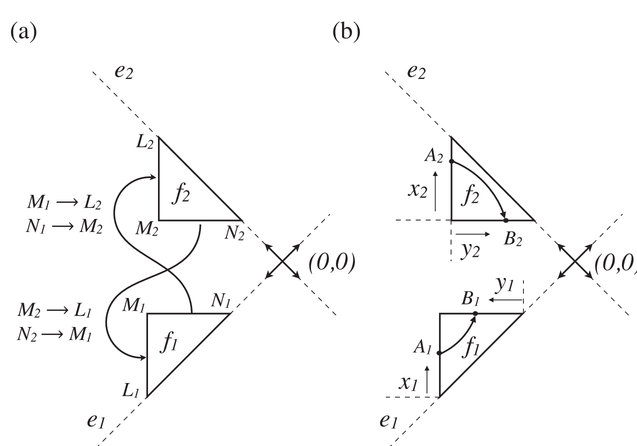

FIG. 11. (a) Symmetry reduced space (fundamental domain) of the coupled system after reducing the symmetries shown in Figure 10. f_1 is the mapping from the left edge of state 1 to its top edge and f_2 is the mapping from left edge of state 2 to its bottom edge. (b) Definition of x_1, x_2, y_1, and y_2 on the edges of the states in the symmetry reduced space.

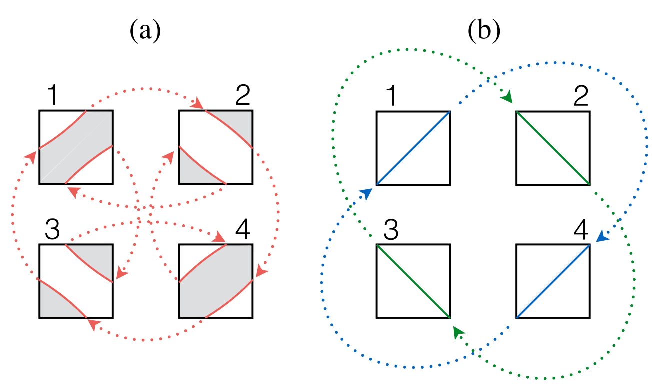

FIG. 13. The trajectories (which are periodic orbits) corresponding to the fixed points in the return maps of Figure 12(b). (a) The unstable fixed point of Figure 12(b) corresponds to two periodic orbits in the unreduced space as shown in red. (b) The fixed point at 0 corresponds to a single periodic orbit shown in blue and the fixed point at dv corresponds to the green periodic orbit. When the initial state of the system lies in the gray region (shown in (a)), the system settles down to an in-phase locking state, and otherwise to an anti-phase locking state.

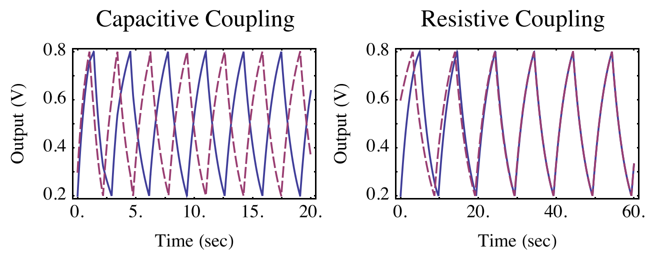

FIG. 14. Capacitive coupling leads to anti-phase locking and resistive coupling leads to in-phase locking in case of symmetric D-D coupled oscillators. The solid and dashed lines represent output of the two oscillators.

Similarly, an implicit equation for f_2: $x_2 \rightarrow y_2$ can be written as

$$\left(\frac{k_\beta + x_2}{k_\beta - y_2}\right) = \left(\frac{dv - x_2}{dv - y_2}\right)^{\frac{\beta+2\alpha}{\alpha+2\beta}}, \tag{23}$$

where $k_\beta = \frac{\beta}{\beta+2}$.

Equations (22) and (23) can be solved numerically to obtain the steady state orbits of the system.

Proposition 1: *Existence of stable periodic orbit and sufficient condition for stable anti-phase locking in symmetric D-D coupled oscillator system:* For $\beta > \alpha > \frac{2dv}{1-dv} = \frac{dv}{v_l}$, i.e., $\frac{g_{dm}}{g_c} > \frac{c}{c_c} > \frac{2dv}{1-dv} = \frac{dv}{v_l}$ the coupled symmetric and identical system has only two steady state locking orbits—in-phase and anti-phase. Further, the in-phase locking is unstable and the anti-phase locking is stable.

Proof: The proof can be divided in two steps-(a) There are only two fixed points of f-at 0 and at dv, and (b) $f'(0) > 1$ and $f'(dv) < 1$ which implies that the in-phase locking is unstable and anti-phase locking is stable.

The first part is proved as follows:

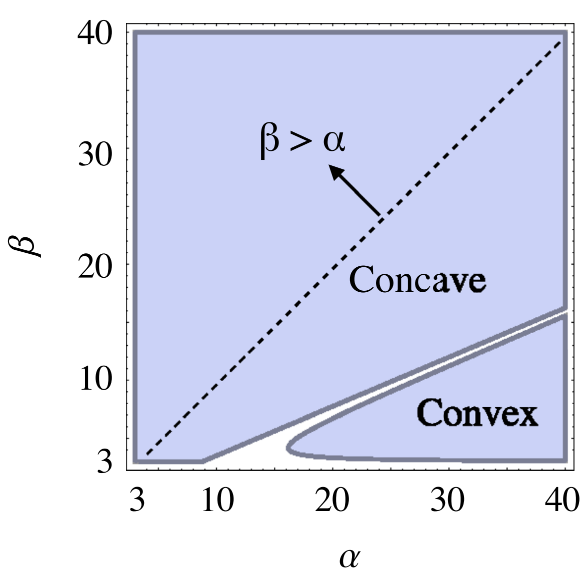

FIG. 15. Return map type for the symmetric D-D case in the parametric space, $\beta \times \alpha$ for $v_l = 0.2$ and $v_h = 0.8$. We can clearly see that for $\beta > \alpha$ the return map is concave and anti-phase locking is stable. Also when the coupling is more resistive, the return map becomes convex with stable in-phase locking. The region between concave and convex return map is the region with S-shaped return map with both stable in-phase and stable anti-phase locking.

As λ_1 and λ_2 are negative, $x_1 > y_1$ and $dv - x_2 > dv - y_2$. And as $\beta > \alpha > \frac{2dv}{1-dv}$, $\frac{\alpha+2}{\beta+2}\frac{\beta}{\alpha} > 1$ and $\frac{\beta+2}{\alpha+2}\frac{\alpha}{\beta} < 1$. Also $\beta > \frac{2dv}{1-dv}$ implies $k_\beta > dv > y_2$. This gives us the following inequalities:

$$\left(\frac{2v_h - x_1}{2v_l + y_1}\right) \geq \left(\frac{x_1}{y_1}\right) \tag{24}$$

$$\text{and } \left(\frac{k_\beta + x_2}{k_\beta - y_2}\right) \leq \left(\frac{dv - x_2}{dv - y_2}\right), \tag{25}$$

where the equality holds at the end points, i.e., at $x_1 = 0$ and $x_1 = dv$ for (24) and at $x_2 = 0$ and $x_2 = dv$ for (25). At any fixed point for the return map f, $x_1 = y_2$ and $y_1 = x_2$ and Eqs. (24) and (25) should be consistent with these fixed point equations. Substituting $x_1 = y_2$ and $y_1 = x_2$ in (24) and (25), we get

$$dv - ((dv - y_1) + y_2) + \frac{2(dv - y_1)y_2}{dv + k_\beta} \geq 0, \tag{26}$$

$$dv - ((dv - y_1) + y_2) + \frac{(dv - y_1)y_2}{v_h} \leq 0. \tag{27}$$

These equations are consistent only when

$$(dv - y_1)y_2 \frac{2}{dv + k_\beta} \geq (dv - y_1)y_2 \frac{1}{v_h}, \tag{28}$$

which in turn can be true only at the end points, i.e., $y_1 = 0$ or $y_1 = dv$, because $k_\beta < 1$. It can be confirmed that this is indeed the case by inspection of Figure 11.

The second part of the proof is proved by calculating $f'(0) = f_1'(0) \cdot f_2'(0)$. $f_1'(0)$ and $f_2'(0)$ are calculated from (22) and (23) as

$$f_1'(0) = \left(\frac{v_l}{v_h}\right)^q, \tag{29}$$

$$f_2'(0) = \frac{k_\alpha + dv}{k_\alpha - dv} = \frac{k_\alpha + v_h - v_l}{k_\alpha - v_h + v_l} > \frac{v_h}{v_l}, \tag{30}$$

where $q = \frac{\beta+2}{\alpha+2}\frac{\alpha}{\beta} < 1$ and $k_\alpha = \frac{\alpha}{\alpha+2}$. Also $\alpha > \frac{2dv}{1-dv}$ implies $k_\alpha > dv$. Hence,

$$f'(0) = f_1'(0) \cdot f_2'(0) > \left(\frac{v_h}{v_l}\right)^{1-q} > 1. \tag{31}$$

And as f has no other fixed points between 0 and dv and f is continuous, $f'(dv) < 1$. Hence, proved.

It should be noted that this condition is not a strict bound but rather provides key design insights when a particular form of coupling (anti-phase) is sought.[1]

A. Capacitive, resistive coupling and bistability

The two extreme cases of purely resistive and purely capacitive coupling are of interest. In case of coupling using only a capacitor, the symmetric and identical coupled system always has a stable anti-phase and an unstable in-phase locking. This is because in case of purely capacitive coupling,

$\beta \to \infty$ and so $\beta > \alpha$ for all finite α. Even in practical cases where some parasitic resistance is included in parallel with the coupling capacitor,[2] β is typically much larger than α. Such anti-phase locking matches well with recent experimental findings of capacitively MIT coupled oscillators as discussed in Ref. 1. In case of coupling using only a resistor, the symmetric and identical coupled system will have a stable in-phase and an unstable anti-phase orbit, as can be predicted from Figure 15 for $\alpha \to \infty$. Time domain simulations of the coupled systems with purely capacitive and purely resistive coupling are shown in Figure 14. The parameter values for capacitive coupling are $\alpha = 5$ and $\beta = 6$ and those for resistive coupling are $\alpha = 13.1$ and $\beta = 3.6$.

Figures 12(b) and 12(c) show cases when $\beta < \alpha$. In the intermediate case when the return map transitions from concave to convex, the system goes through a state where both in-phase and anti-phase locking are stable with one unstable fixed point in between (Figure 12(b)). In Figure 15, the two regions for concave and convex return map can be clearly seen. They are separated by a thin region which represents the case of bistability. Figure 16 shows the time domain simulation waveforms of oscillator outputs for $\beta = 3.6$ and $\alpha = 13.1$. We note that the initial voltage of the first oscillator is 0.2 V and depending on the initial voltage of the second oscillator, the system can either lock in phase or out of phase. These design parameters correspond to a bistable system of the kind shown in Figure 12(b), and hence the final steady state locking is in-phase or out-of-phase depending on the initial phase of the system. When the initial phase (or output voltage) of oscillators are close to each other (represented by gray region in Figure 13(a)) the system locks in-phase, and when they are far the system locks out-of-phase for the same circuit parameters.

VI. ASYMMETRIC D-D COUPLED OSCILLATOR DYNAMICS

Let us now investigate the case of D-D oscillator dynamics where the two oscillators are identical but the pull-up and pull-down devices are non-identical thereby giving rise to asymmetric charging and discharging rates. As the oscillators are identical, $\beta_{11} = \beta_{21} = \beta_c$ and $\beta_{12} = \beta_{22} = \beta_d$, where subscripts c and d stand for charging and discharging. The

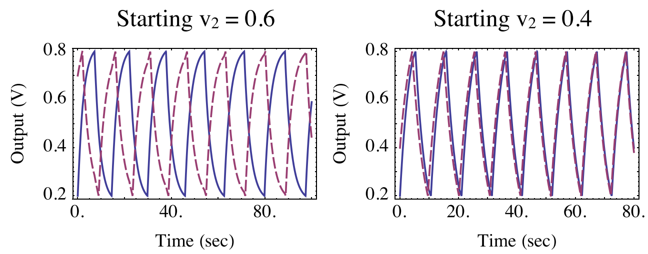

FIG. 16. Simulation waveforms showing the dependence of final locking to the initial state of the system in the intermediate case of Figure 12(b) when the return map is S-shaped. The solid and dashed lines represent the two oscillators. Initial $v_1 = 0.2$ V in both cases, but the system locks in-phase when initial $v_2 = 0.4$ V, and anti-phase when initial $v_2 = 0.6$ V. With reference to Figure 13, the initial point (0.2, 0.4) lies in the gray region and the point (0.2, 0.6) lies outside the gray region in conduction state 1.

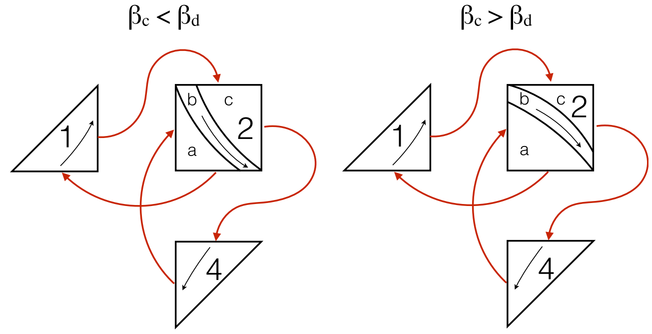

FIG. 17. Symmetry reduced space in the asymmetric D-D configuration with $\beta_c > \beta_d$ (left) and $\beta_c < \beta_d$ (right). Such configuration will have only a single symmetry. The flow matrices in the four conduction states are not equal and hence states cannot be represented in a single combined state space with a single fixed point as was done in the symmetric D-D case ($\beta_c = \beta_d$).

symmetry of the system (due to the identical oscillators) can be seen in the flows of the states. Flows of conduction states 1(00) and 4(11) are mirror images about the diagonal $x = y$ and the flow in conduction state 2(10) is equivalent to the flow in state 3(01) with axes x and y interchanged. This symmetry is also shown in the transitions between states. The system can be expressed after reducing the symmetry as in Figure 17. For $\beta_c < \beta_d$, two kinds of cycles are possible in the regions $1 \to 2b \to 1$ and $1 \to 2c \to 4 \to 2a \to 1$. To find the fixed points of the system, we draw the return map with the bottom edge of state 1 as the Poincare section. Let f_1 be the mapping from bottom edge of conduction state 1 to its right edge, and f_{2a}, f_{2b}, and f_{2c} are the mappings between edges in conduction state 2 as shown. Also, let $f_1(x'_k) = x_k$ and $f_{2b}(0) = y_k$ as shown in Figure 18. For small asymmetries, the flows remain monotonic and also $y_k > x'_k$. Because it is a symmetry reduced space, we consider the first return map for trajectories of the type $1 \to 2c \to 4 \to 2a \to 1$ and the second return map for trajectories of the type $1 \to 2b \to 1$. Then, the return map f is given by

$$f(x) = \begin{cases} f_1 \circ f_{2c} \circ f_4 \circ f_{2a}(x), & 0 \le x < x'_k \\ f_1 \circ f_{2b} \circ f_1 \circ f_{2b}(x), & x'_k \le x < dv. \end{cases} \tag{32}$$

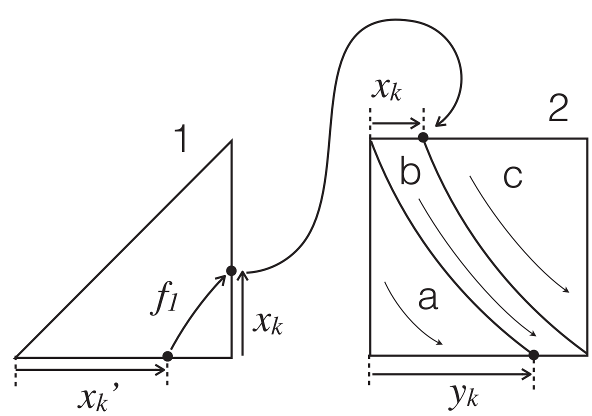

FIG. 18. Diagram of symmetry reduced state spaces for conduction states 1 and 2 in the asymmetric D-D configuration. In conduction state 2, the top-left corner does not map to the bottom-right corner as was the case in the symmetric D-D case. The width of this middle region where the flow maps the top edge to the bottom edge is defined using x_k and y_k.

Proposition 2: *Sufficient conditions for existence of stable periodic orbit in asymmetric D-D coupled oscillator system:* If in a D-D asymmetric coupled oscillator system the asymmetries are small enough such that the flows are monotonic and $y_k > x'_k$, then the following are true about the return map f on the bottom edge of state 1:

(a) f is continuous
(b) $f'(0) > 1$ for $\beta_c > \alpha$ and $\beta_d > \alpha$
(c) f has one fixed point at 0 and at least one in the interval $x'_k < x < dv$ at, say, x_f
(d) Either the fixed point at x_f is stable, or there exists a stable fixed point at x'_f, where $0 \le x'_f < x_f$

Proof: (a) The return map is separately continuous in intervals $[0, x'_k)$ and $(x'_k, dv]$ as it is a composition of mappings of continuous flows. The continuity of f at x_k can be established by considering two points close to x'_k on either side. From (32), we can see that $f(x'_{k+}) = f(x'_{k-}) = y_k$, and hence, f is continuous at x_k.

(b) It can be proved by similar procedure as adopted before in *Proposition 1* that $f'(0) = f'_1(0) \cdot f'_{2c}(dv) \cdot f_4(0) \cdot f'_{2a}(0) > 1$ for $\beta_c > \alpha$ and $\beta_d > \alpha$.

(c) The fixed point at 0 can be seen clearly in the flow diagram. In interval $x'_k < x < dv$, the fixed points of first return $f_1 \circ f_{2b}$ will also be the fixed points of second return (which is f), but not the other way around. Now $f_1 \circ f_{2b}(x'_k) = dv$ and $f_1 \circ f_{2b}(dv) = y_k$. As $f_1 \circ f_{2b}$ is continuous, and hence decreasing, in the interval $x'_k < x < dv$, there exists a fixed point for $f_1 \circ f_{2b}$, and hence for f, in the interval $x'_k < x < dv$.

(d) As f is continuous and has fixed points at 0 and x_f, one of these two should be stable if there is no other fixed point in between 0 and x_f. If they both are unstable, then a stable fixed point exists in the interval $(0, x_f)$. Hence, proved.

Figure 19 shows a representative return map for the asymmetric D-D configuration. The poincare section chosen in the symmetric D-D case was the left edge of conduction state 1. Due to symmetry, the left edge of conduction state 1 is same as the bottom edge of conduction state 1. Hence, the return maps in the symmetric D-D case can be compared with the return maps in the asymmetric D-D case as if they were drawn on the same edge. Figure 20 shows a comparison of the return maps of a symmetric case ($\beta_c = \beta_d = 60$, $\alpha = 10$) with that of two asymmetric cases ($\beta_c = 50$ and 40, $\beta_d = 60$, $\alpha = 10$). The corresponding time domain waveforms and phase plots are shown in Figure 21. The figure clearly shows that the steady state periodic orbit changes from a diagonal (perfect anti-phase locking) to a butterfly shaped curve (imperfect anti-phase locking) as the asymmetry

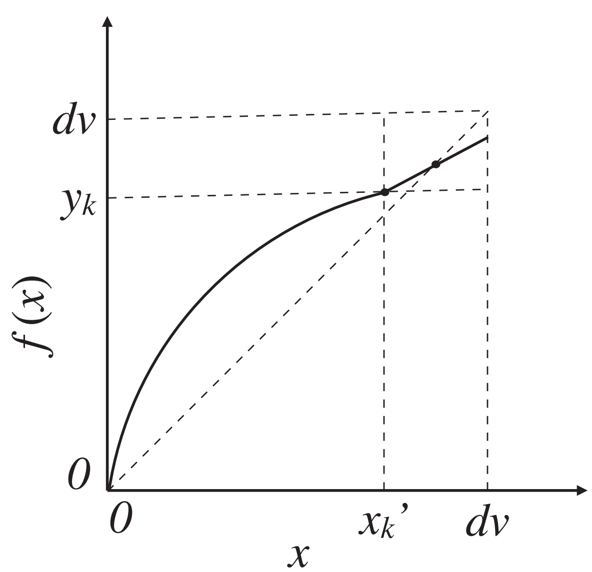

FIG. 19. Representative plot of the return map on the bottom edge of conduction state 1 in the asymmetric D-D case. The fixed point corresponding to anti-phase locking which was at dv in the symmetric case is shifted inside away from dv in the asymmetric case.

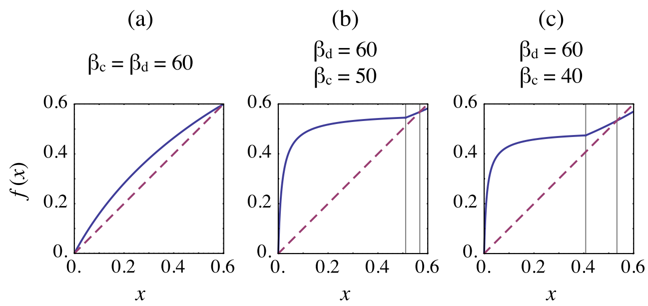

FIG. 20. Comparison of return maps in the symmetric (a) and asymmetric ((b) and (c)) D-D configurations for constant $\alpha = 10$. Both symmetric and asymmetric configurations have a fixed point at 0 corresponding to in-phase locking (which is unstable here as $\beta > \alpha$ condition is satisfied) along with another fixed point, which in symmetric case, is at dv (perfect anti-phase locking) but in asymmetric case shifts away from dv.

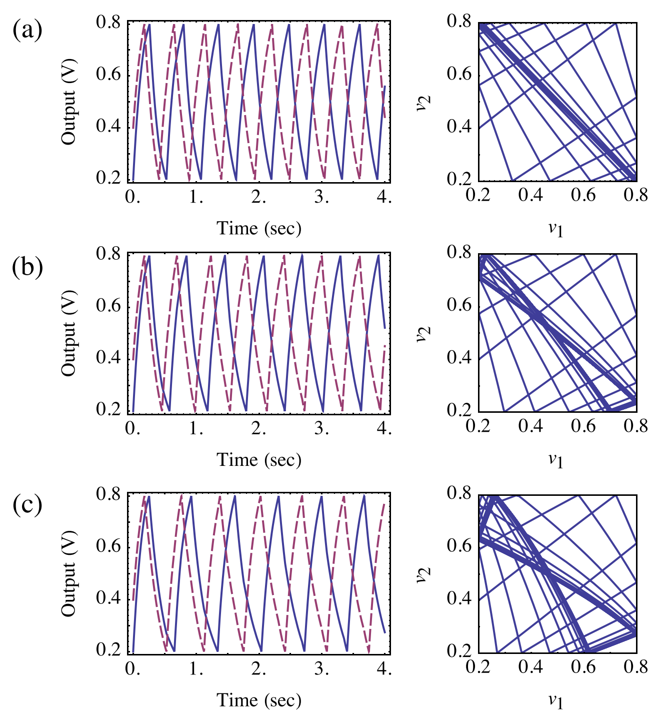

FIG. 21. Time domain waveforms and phase plots corresponding to the configurations in Figures 20(a)–20(c). The steady state periodic orbits can be seen clearly in the phase plots to transform from a diagonal (perfect anti-phase locking) in the symmetric case (a) to a butterfly shaped curve (imperfect anti-phase locking) as the asymmetry increases and the anti-phase fixed point in the return map shifts away from dv.

increases. However, the time domain waveforms for butterfly shaped periodic orbits would still be very similar in appearance to anti-phase locking. The fixed point close to dv in the return map shifts away from dv as the difference between β_c and β_d increases. This trend can be seen in Figure 22 which shows the movement of the anti-phase fixed point with β_d–β_c for fixed $\beta_d = 60$ and $\alpha = 10$. For $\beta_c > \beta_d$, the cycles will be of the type $4 \rightarrow 2b \rightarrow 4$ and $1 \rightarrow 2c \rightarrow 4 \rightarrow 2a \rightarrow 1$, and the return map will have to be drawn on an edge of state 4. The return map in this case will be analogous to the $\beta_c < \beta_d$ case with β_c and β_d interchanged.

VII. D-R COUPLED OSCILLATOR DYNAMICS

In this section, we consider the dynamics of a D-R coupled system. This is of interest because of its ease of fabrication, relaxed conditions for oscillations and already published reports of such coupled oscillatory systems.[2] We consider coupling of identical oscillators and hence, we define $\beta_1 = \beta_2 = \beta$ and $\beta_{s1} = \beta_{s2} = \beta_s$. Unlike the D-D coupled oscillator case, the notion of symmetric charging and discharging does not apply in D-R coupled oscillator case because the circuit by construction is different for charging and discharging. During charging a part of the net charging current charges up the output capacitor whereas the rest of it flows through the pull-down resistance to ground. The process of discharging has no such leakage component. In terms of the conductance ratio β, this can be explained by the fact that the net charging component in the matrix A is $(\beta + \beta_s)$ and it is always greater than the discharging component β_s. However, the flows can still be simplified for analysis as was described in Sec. IV. The simplification assumes that the flows are monotonic in the regions of operation in all four conduction states, but the direction of monotonicity is different from the D-D coupled oscillator case as shown in Figure 6. For our analysis, a particular type of non-monotonicity is allowed in state 2 (and state 3) as shown in Figure 23. Here, the fixed point for conduction state 2 satisfies the condition of oscillation shown in Figure 7, but the

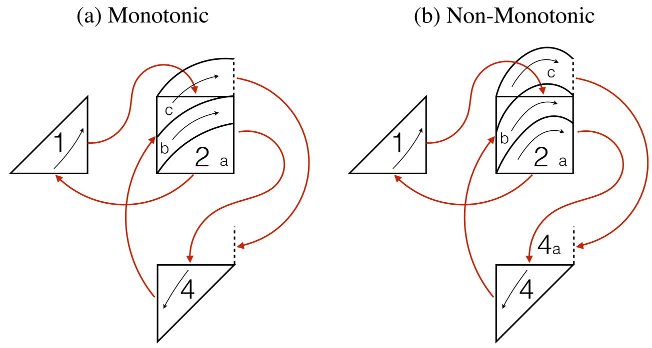

FIG. 23. Symmetry reduced space in the D-R coupled oscillator system. There is only a single symmetry due to identical oscillators.

flow in state 2 as shown in the symmetry reduced space (Figure 23) is non monotonic. We will consider the case of identical oscillators, and following the methodology of the asymmetric D-D case, we can reduce the symmetry of identical oscillators as shown in Figure 23. In this case, two kinds of cycles are possible - $4 \rightarrow 2b \rightarrow 4$ and $4 \rightarrow 2c \rightarrow 4a \rightarrow 4$. To find the fixed points of the system, we draw the return map on the top edge of conduction state 4 as the Poincare section. Let f_4 be the mapping from top edge of state 4 to its left edge, f_{4a} be the mapping from the extended right edge of state 4 to its top edge, f_{2a}, f_{2b}, and f_{2c} be the mappings between edges of state 2. Also let $f_4(x_k') = x_k$ and $f_{2b}(0) = y_k$ as shown in Figure 24. We consider the scenario when the flows of the system are as shown in Figure 23 and $y_k > x_k'$. Because it is a symmetry reduced space, we will have to consider the second return map for cycles of the type $4 \rightarrow 2b \rightarrow 4$ but only the first return map for $4 \rightarrow 2c \rightarrow 4a \rightarrow 4$ type cycles. Then, the return map f is given by

$$f(x) = \begin{cases} f_4 \circ f_{2c} \circ f_{4a}(x), & 0 \le x < x_k' \\ f_4 \circ f_{2b} \circ f_4 \circ f_{2b}(x), & x_k' \le x < dv. \end{cases} \quad (33)$$

Proposition 3: *Sufficient conditions for existence of stable periodic orbit in D-R coupled oscillator system:* If in a D-R coupled oscillator system, the flows are as shown in Figure 23 and $y_k > x_k'$ then the following are true about the return map on the top edge of state 4 in the symmetry reduced state space (Figure 23)

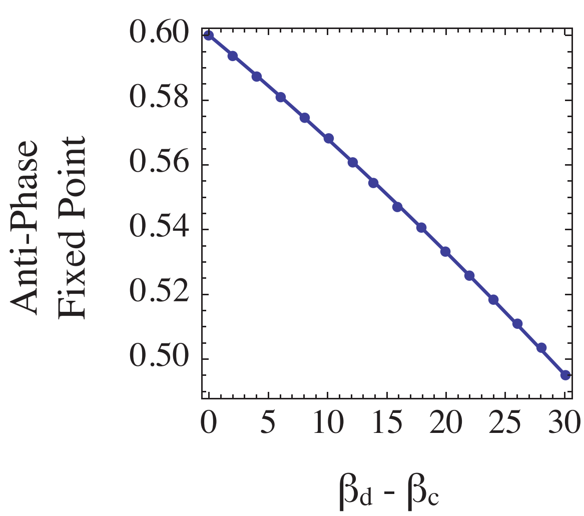

FIG. 22. Numerical simulations illustrating the fixed point close to dv shifts away from dv with increasing difference between β_c and β_d in the asymmetric case.

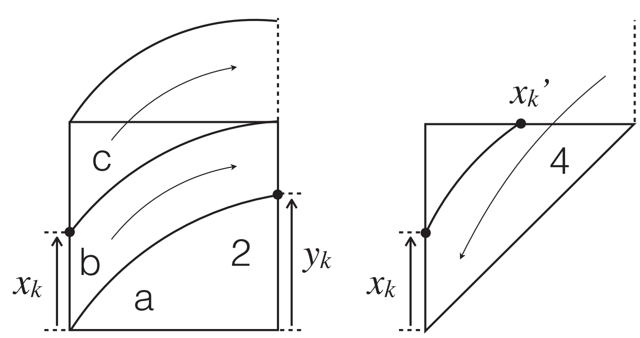

FIG. 24. Symmetry reduced space for the D-R coupled oscillator system in states 1 and 4 with the definition of x_k, x_k', and y_k. f_4 is the mapping from top edge of state 4 to left edge of state 4 and f_{2a}, f_{2b}, and f_{2c} are mappings between edges of state 2 as shown.

(a) f is piece-wise continuous with discontinuity at x'_k. Moreover, $f(x'_{k+}) = y_k$ and $f(x'_{k-}) = dv$.
(b) f has at least one fixed point in the interval $x'_k < x < dv$ at, say, x_f.
(c) f has at least one stable fixed point in the interval $x'_k < x < dv$.

Proof: (a) The argument is the same as in *Proposition 2*. The return map is separately continuous in intervals $[0, x'_k)$ and $(x'_k, dv]$ as it is the composition of continuous flows. From (33), we can see that $f(x'_{k+}) = y_k$ and $f(x'_{k-}) = dv$.

(b) In the interval $x'_k < x < dv$, the fixed points of the first return map $f_4 \circ f_{2b}$ will also be the fixed points for its second return map (which is f). Now $f_4 \circ f_{2b}(x'_k) = dv$ and $f_4 \circ f_{2b}(dv) = y_k$. As $f_4 \circ f_{2b}$ is continuous (and hence decreasing) in this interval, there exists a fixed point for $f_4 \circ f_{2b}$, and hence f, in the interval $x'_k < x < dv$.

(c) As $f(x'_{k+}) = y_k$, f is continuous in the interval $x'_k < x < dv$ and f has a fixed point at x_f where $x'_k < x_f < dv$, hence either the fixed point at x_f is stable or there exists another fixed point in the interval $x'_k < x < x_k$ which lies in $x'_k < x < dv$. Hence proved.

Figure 25 shows the return map f on the top edge of state 4 for the D-R coupled oscillator system for varying β_s. The return maps in the figure have a single stable fixed point at x_f in the interval $x'_k < x < dv$. The movement of the fixed point x_f with β_s is shown in Figure 26.

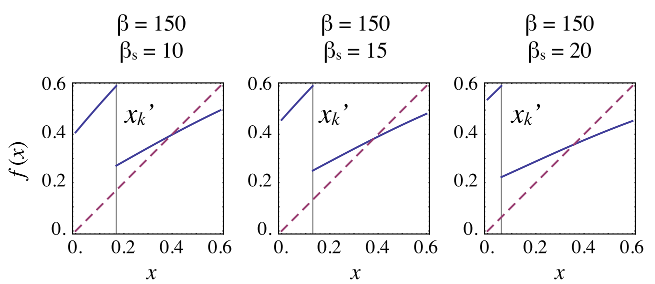

FIG. 25. Return map on the top edge of state 4 for the D-R coupled oscillator system for $\alpha = 1$, $\beta = 150$ and βs values of 10, 15, and 20.

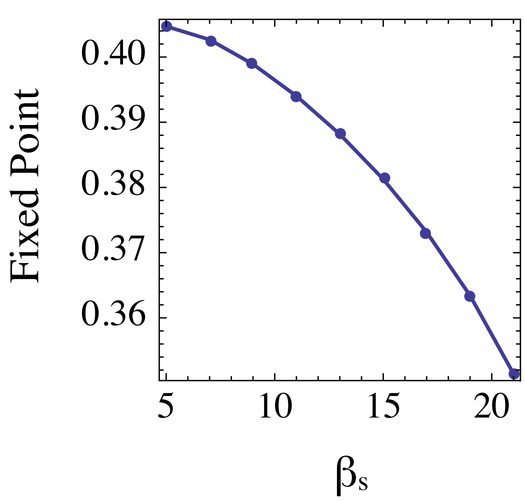

FIG. 26. Movement of the fixed point x_f for fixed $\alpha = 1$, $\beta = 150$ and varying β_s for the return map on the top edge of state 4 for the D-R coupled oscillator system.

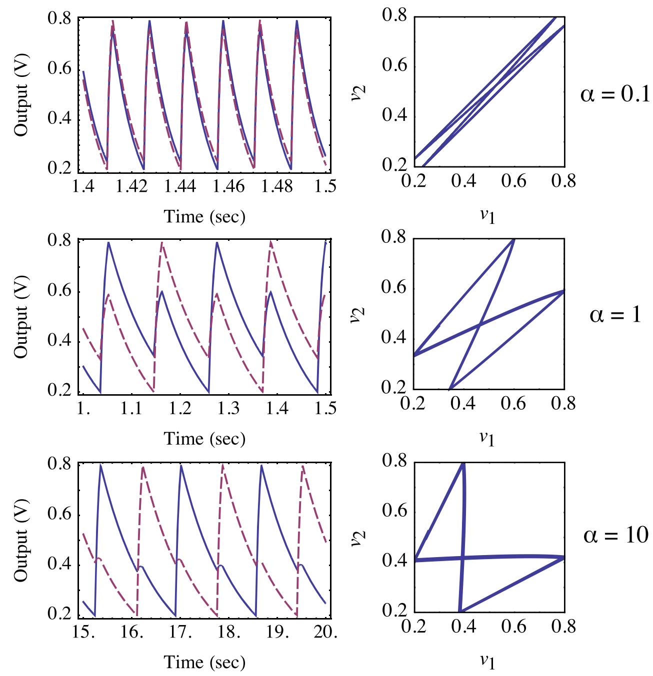

FIG. 27. Steady state waveforms and phase trajectories for the D-R coupled oscillator system with $\alpha = 0.1$ (top), $\alpha = 1$ (middle), and $\alpha = 10$ (bottom). The solid and dashed lines represent the two oscillators.

Another important design consideration for the coupled oscillator system, is the role of the coupling circuit on the overall system dynamics, as is seen in Figure 27. We note that as the value of α increases the phase diagram in the $v_1 \times v_2$ plane shows strong sensitivity. In particular, for low values of α, the system shows in-phase locking. As α increases (for intermediate value of α), the butterfly shaped phase plot widens and the system exhibits a non-monotonic decrease in the output voltages, v_1 and v_2 from v_h to v_l. This can also be seen in the time domain waveforms where the output voltages first decrease to an intermediate voltage, then increase and again decrease; clearly demonstrating four possible conduction states (MM, MI, IM, and II) in both phase and time domain plots. Finally, for high values of α the butterfly in the phase plot opens even further, thus making the decrease of output voltages from v_h to v_l more monotonic and the system tends to anti-phase locking, as exhibited in both phase and time (Figure 27).

VIII. OTHER POSSIBLE DYNAMICS AND INSTABILITY

For sustained oscillations, the fixed points of the flows should lie outside the region of operation in all the system configurations that have been discussed. When the fixed points lie inside the region of operation, the oscillations stop and the system settles down at the fixed points. In the case of sustained oscillations, the symmetric D-D coupled oscillator system always has a monotonic flow and a stable periodic orbit. For the other two cases of asymmetric D-D and D-R coupled oscillator systems, we have explored sufficient conditions for stability, namely, the existence of monotonic flows and $y_k > x'_k$. But when these conditions are not satisfied, the system might not have a stable periodic orbit. For asymmetric D-D coupled oscillator case, the conditions of monotonicity and $y_k > x_k'$ hold for small asymmetries, i.e., small difference between β and β_s. For large asymmetries, these conditions may be violated and more complex system dynamics evolve. On the other hand, in the D-R coupled oscillators, the conditions of monotonicity and $y_k > x_k'$ hold for large differences in β and β_s, and for smaller differences, the conditions may be violated. In either case, when these conditions do not hold, the system might have periodic orbits with more than 4 transitions per period or no periodic orbit with irrational rotation numbers.[16]

IX. EXPERIMENTAL VERIFICATION

An MIT device can be realized using VO_2 (Vanadium dioxide) which exhibits unique electronic properties like metal-insulator phase transitions. VO_2 has been shown to undergo abrupt first order metal-to-insulator and insulator-to-metal transitions with up to five orders of change in conductivity[19] and ultra-fast switching times.[17] The time constants using discrete circuit elements are usually high[1] due to the effect of parasitic elements. Faster time constants of sub-100 fs have been experimentally demonstrated in monolithically integrated correlated material oscillator devices[20,21] Moreover, it has been reported that the phase transition (i.e., IMT and MIT) in correlated materials like vanadium dioxide (VO_2) fundamentally occurs on extremely fast time-scales ($\sim$75 fs).[22,23] The expected time constants for electrically induced phase transition in scaled VO_2 devices have also been investigated in Ref. 17. These phase transitions occur at time scales that are a few orders of magnitude smaller than the RC time constants in the oscillator circuit (Figure 29). Transitions have been shown to be electrically driven, thermally driven or a combination thereof. Recent work shows that for such a transition, a metallic filament structure is formed which acts as a conduction pathway in the low resistance state of VO_2.[24] Also, a series circuit of VO_2 with a resistive pull down network has been shown to exhibit self-sustained electrical oscillations[2] when conditions of oscillations as described above are met. Moreover, two such relaxation oscillators can be electrically coupled to produce synchronized oscillations.[2] For experimental validation, we apply our models of coupled relaxation oscillators on a system of two coupled VO_2 oscillators. Figure 28 shows a schematic representation of the coupled circuit with a parallel resistance (R_C) and capacitance (C_C) as the coupling circuit. Frequency domain results of this system have been previously reported[2] showing a close match between experiments and theoretical results of a D-D model; and are not reproduced here. Using the D-R model developed in this paper, we obtain close match in the time-domain and phase plots of the oscillator system as well. With proper calibration of the system parameters, the D-R model described above shows very close qualitative match with experimental results. One such experimental result has been shown in Figure 29 along with model prediction. This validation of the proposed models enables further design of experiments. It further models and explains both qualitative and quantitative the role of the system design parameters on the rich synchronization dynamics.

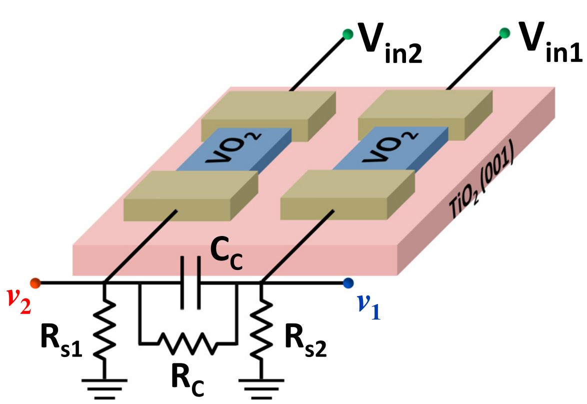

FIG. 28. Schematic of the experimental setup of coupled VO_2 oscillators, with series resistances R_{s1} and R_{s2}, respectively, coupled using a parallel R_C–C_C circuit.

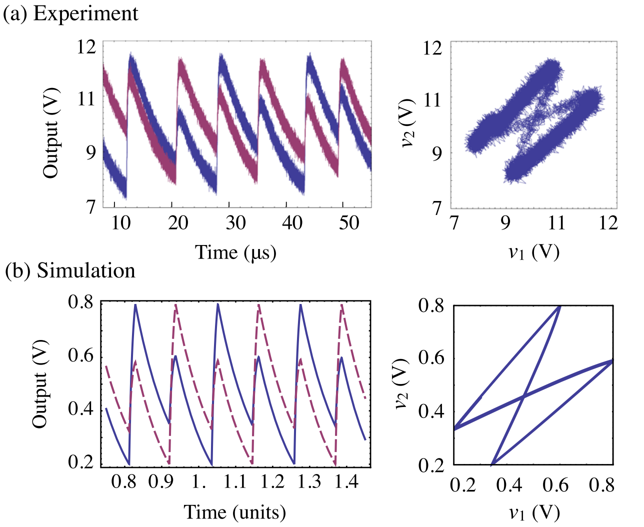

FIG. 29. Experimental and simulated time domain waveforms in the steady state and phase plots for a parallel R_C–C_C coupled oscillator system. The D-R coupled relaxation oscillator model is used for model development and simulation. The two waveforms show close match and validate the model prediction.

X. CONCLUSIONS

This paper presents a model study of the synchronization dynamics of a pair of identical and electrically coupled relaxation oscillators when physically realized using MIT devices. Experimental realization of such devices[1,2] has motivated the study of their dynamics, with emphasis on phase synchronization, locking conditions and potential programmability of the phase relations using electrical means. We investigate the case of a purely MIT based oscillator (D-D) and that of a hybrid oscillator composed of an MIT device and a passive resistance (D-R configuration). We show through numerical and analytical techniques, validated against experimental results, the existence of out-of-phase locking (in purely capacitive coupling), in-phase locking (in purely resistive circuits) and the possibility of bistable circuits (for intermediate values of R and C). This opens new paradigms for realizing associative computing networks using coupled oscillators by enabling model studies of such physically realizable circuit elements.

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